Simple Growth Models

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Abstract

Notes for G8412. I describe a set of simple growth models and highlight the role that political action and decision making plays in each.

1 Solow Model

One of the simplest models of economic growth—and one that forms the basis for many other theoretical and many empirical models—is Solow's neoclassical growth model. It works like this.

1.1 The Economy

First we assume that the outcome of the economy can be described by some production function, F. This function tells us how much income, Y, is generated in a given year, for a certain amount of inputs, such as capital, K, or labor, L. These inputs are called the **factors of production**. The productive side of the economy at a given point in time—say at time t, is given then by:

$$Y_t = F(K_t, L_t)$$

Some assumptions are typically made about F. One, evidently is that F is increasing in K and L. However, we also assume that the *marginal* products of K and L are decreasing: this means that if you keep labor fixed then you get less bang for each additional buck of capital. And if you keep capital fixed, you get less out of each additional worker you add to the economy. These conditions are written formally as $F_K > 0, F_L > 0$, $F_{KK} < 0, F_{LL} < 0$.

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Another assumption is that F exhibits **constant returns to scale**: this means if you increase *all* the inputs by some factor, then you increase the output by that factor.

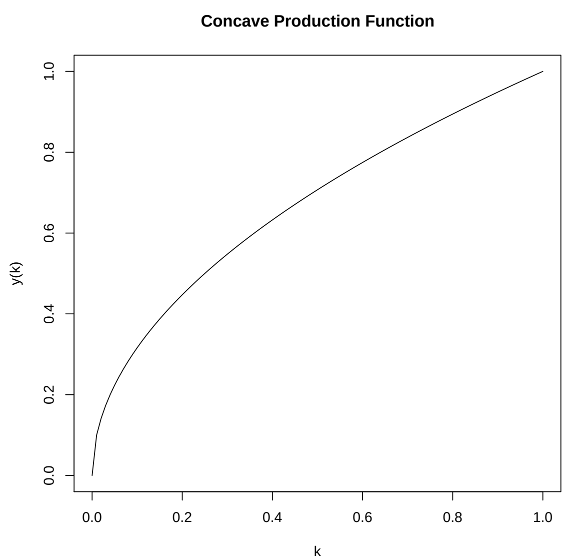
A final set of conditions typically used—called the **Inada conditions**—require that the marginal products of a factor be extremely high when there is little of that factor, and that they approach zero when there is lots of the factor.

If these conditions are satisfied, we call the model **neoclassical**.

One useful feature of the assumption of constant returns to scale is that it allows us to express output per capita as a simple function of capital per capita. To see this, note that the constant returns condition implies that $\frac{Y_t}{L_t} = \frac{1}{L_t}F(K_t, L_t) = F\left(\frac{K_t}{L_t}, 1\right)$. We can then make things a little easier by defining everything in per capita terms, defining the function $f\left(\frac{K_t}{L_t}\right) = F\left(\frac{K_t}{L_t}, 1\right)$ and defining the variables $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$. We then have:

$$y_t = f(k_t)$$

It is easy to see that f is increasing in k but has diminishing marginal returns in k. If we graph it, it will look something like this:



This figure shows the amount of per capita output you can get from different levels of per capita capital, given all other features of your economy. Note that because y_t depends only on k_t, changes in y—and hence economic growth rates—depend only on changes in k, that, is, on the rate of accumulation of capital. We will now turn then to consider the accumulation of capital in this economy.

1.2 Investment

A key driver of growth in this economy results from changes in the capital stock. If the capital stock grows over time, so will output.

To describe changes in capital we need a *law of motion.* We assume simply that capital in one period is equal to capital in the last period, less any depreciation that occurs—which we assume to be a constant share, δ, of last period's capital— plus any new investments—which we assume to be determined by the amount of last period's production that was saved, rather than consumed, according to a fixed savings rate, s. Hence:

$$K_{t+1} = K_t + s \times F(K_t, L_t) - \delta \times K_t \tag{1}$$

We assume that the amount of labor is fixed (although it's a good exercise to solve a version where $L_{t+1} = (1+n) \times L_t$, and observe the effects of the growth rate of labor on the growth rates of per capita income). We can then simplify Equation 1 by dividing across by L_t. Using the fact that $L_t = L_{t+1}$ we then have:

$$k_{t+1} = k_t + s \times f(k_t) - \delta \times k_t \tag{2}$$

This is an expression for next year's per capita capital given this year's per capita capital. We can see quickly that k_{t+1} is large when s is large, and that it is low when δ is large. All this means is that the more you save this period and the less is destroyed, the more you have next period. No surprises there. It is also the case that k_{t+1} is larger whenever k_t is large. This does not however mean that economies always grow.

1.3 Growth of (per capita) Capital

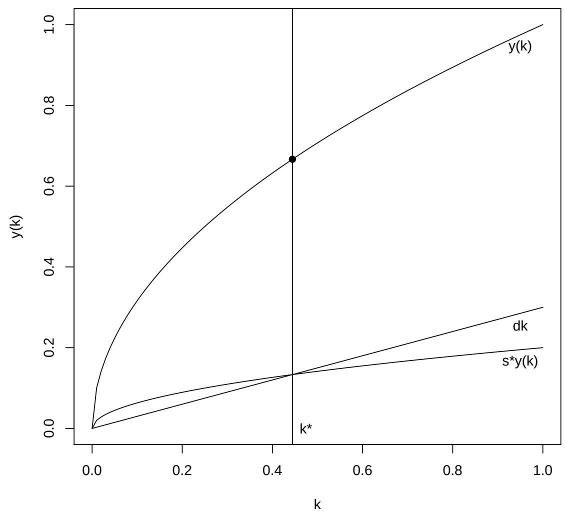
When we are interested in growth, we care not about whether k_{t+1} is large when k_t is large, but whether $k_{t+1}$is larger than k_t. That is, whether:

$$k_{t+1} - k_t = sf(k_t) - \delta k_t > 0 \tag{3}$$

Clearly this is true if and only if:

$$sf(k_t) > \delta k_t \tag{4}$$

This condition has a simple interpretation. The left hand side of the expression is a curve, that, while increasing in k_t, is increasing at a diminishing rate. For low values of k_t it is increasing very quickly, but at high levels it is increasing very slowly. The right hand side however, describes a straight line, increasing at constant rate δ. These features imply that for one (and only one) value of k_t — we'll call it k^*— do we have $sf(k_t) = \delta k_t$. The logic is illustrated below.



A lot hinges on k^*. For all values below k^*, Equation 4 holds and the capital stock increases. But if $k_t > k^*$ then capital stock actually falls—the depreciation effects overpower the savings and productivity effects. If however $k_t = k^*$, then $k_{t+1} = k_t$, and capital stays constant, and so economic output stays constant, and so economic growth is zero. For this reason we call this level of capital the **steady state** capital level. It is *steady* in two senses—first of all it is an equilibrium—*if you get there you stay there*; but second it is a stable equilibrium— *if, for some reason, you shift away from there, you'll move back there again.*

1.3.1 Properties of the Steady State

We can learn more about the properties of k^* by studying Equation 4 more. In particular we can see quickly that the higher is δ, the steeper is the right hand side and the more quickly do the curves in Figure 2 cross—this implies that k^* is lower in these cases. Similarly the more quickly $f(k_t)$ flattens out and the lower is s, the flatter is the left hand side and so, again the lower is k^*.

The model predicts then that, *ceteris paribus*, economies are wealthier (in per capita terms) when:

- The savings rate is high

- The depreciation is low
- Productivity is high

1.3.2 Properties of the Growth Rate

Finally we can make a statement about how fast or how slow income grows when the economy is not at its steady state. Manipulating Equation 3 gives an expression for the growth rate as a function of k_t.

$$g(k_{t+1}) \equiv \frac{k_{t+1} - k_t}{k_t} = s\frac{f(k_t)}{k_t} - \delta \tag{5}$$

We can see easily that $g(k_{t+1})$, like k^* itself, is increasing in s, but decreasing in δ.

It is also the case that $g(k_{t+1})$ is decreasing in k_t: that is richer countries grow more slowly. From Equation (5) it is clear that this feature depends on whether $\frac{f(k_t)}{k_t}$—output per unit capital— is decreasing or increasing in k_t. But from diminishing marginal returns we know that *per capita output per unit capital in this model is decreasing.*[1]

The model predicts then that, *ceteris paribus*, economies grow faster (in per capita terms) when:

- The savings rate is high
- The depreciation is low
- Capital (and hence income) is low

The final statement—that growth rates are highest when income is lowest—is the convergence hypothesis. It states that poor economies should grow more quickly and rich countries should grow more slowly, and possibly shrink. Note that this is a statement about the tendency of countries to converge to the steady state rather than a statement directly about the distribution of capital across countries. (Of course if the production function or other features are different in different countries then while each country may still converge they will not all converge to the same steady state).

[1] There are many ways to see this. One is to note that $\frac{f(k_t)}{k_t}$ also describes the output of labor per unit capital. This is increasing in labor per unit capital and hence decreasing in capital per unit labor. That is: $\frac{f(k_t)}{k_t} = \frac{F(\frac{K_t}{L_t},1)}{\frac{K_t}{L_t}} = \frac{\frac{1}{L_t}F(K_t,L_t)}{\frac{K_t}{L_t}} = \frac{1}{K_t}F(K_t, L_t) = F(1, \frac{L_t}{K_t})$, which is increasing in $\frac{L_t}{K_t}$ and hence decreasing in $\frac{K_t}{L_t} = k_t$.

At the steady state however there is zero growth. Hence one of the main predictions of the Solow growth model is that *in the long run there is no growth*. Growth can be introduced in a somewhat exogenous manner by assuming for example that the function $F(K, L)$ changes over time, because of exogenous technological progress. But this is less a result of the model and more of a statement about how technology evolves.

What of politics?

- In this model **government** might improve welfare by encouraging (or discouraging) savings so that the economy hits on a well defined optimal savings rate. (Here the savings rate is exogenous but clearly there is some optimal "middling" level; since if s is too low there is no growth and if it is too high there is no consumption).
- Improvements in welfare can also be achieved by improving productivity; or reducing depreciation.

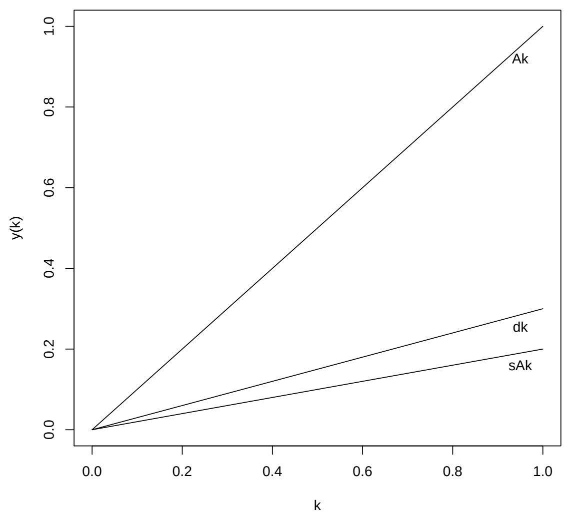
2 An AK Model

The eventual drop in growth rates to zero in the Solow model results from the fact that the driving force for growth is capital accumulation. Since there are diminishing marginal returns to capital, the engine of growth becomes weaker and weaker over time. Simple (if perhaps poorly motivated) models that get beyond this aspect of the Solow model, assume that output per person is proportional to the capital stock.

To see the logic working, return to Equation (5) but assume that the function $f(k_t)$ is given simply by $f(k_t) = Ak_t$. We then have:

$$g(k_{t+1}) = s\frac{Ak_t}{k_t} - \delta = sA - \delta \tag{6}$$

In this case, everything depends on whether sA is greater than δ. If it is, then there is constant growth (and no transitional dynamics to any steady state). If sA is less than δ then the capital stock eventually gets eaten away and spread thin to the point where the economy is run into the ground. This simple model then provides a logic something like that of a **poverty trap**. Depending on a nation's levels of s, A, and δ, different countries can find themselves on paths of constant growth or paths of constant decline (until they hit rock bottom). An example of a shrinking economy is shown in the next figure.



What role for government? Again government could improve welfare by altering savings, depreciation and productivity parameters so that equilibrium growth levels are positive rather than negative.

3 A Simple Poverty Trap Model

While the last model implies that economies fall into one of two types (ever growing or ever shrinking), which of the two types results depends on s, A,and δ. These are structural features of the economy rather than descriptions of its present point on some dynamic path. A richer notion of a trap would have *multiple equilibria* that "coexist" for a given economy—that is, for given levels of s, A, and δ.

Here is one example of such a situation. In this case we assume that there is a minimum amount of per capita consumption, $\underline{c}$, that is needed before individuals are able to save and invest in new capital. Insofar as possible they aim to achieve $\underline{c}$. Only once they surpass this level do they save some share of surplus earnings.

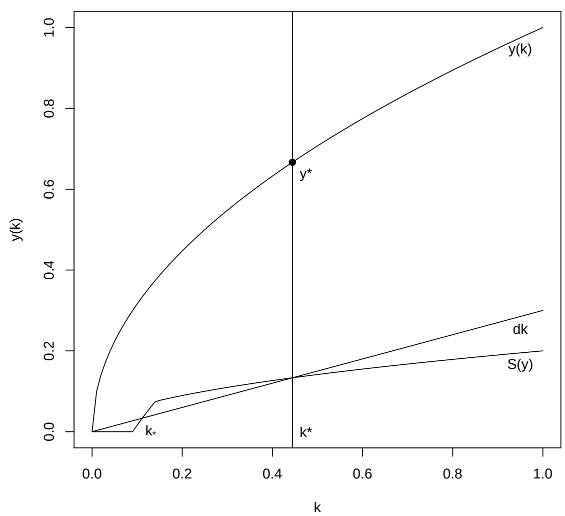
In this case, savings is given by:

$$S(y) = \begin{cases} 0 & \text{if} \quad y \leq \underline{c} \\ y_t - \underline{c} & \text{if} \quad \underline{c} \in \{(1-s)y, y\} \\ s \times y & \text{if} \quad \underline{c} \leq (1-s)y \end{cases}$$

(Note here we use S for the function and s for the savings rate.) The net change in capital (using a version of Equation 3) is given by:

$$k_{t+1} - k_t = S(y_t) - \delta k_t$$

Clearly in this case, the capital stock is declining whenever $y \leq \underline{c}$. It may also be declining in the second range if $y_t < \underline{c} + \delta k_t$. In this case a poor country, will become poorer. However, a country with the same parameters, but at a higher income level may also observe a rise in income—in particular if $\frac{\underline{c}}{(1-s)} < y_t < y^*$, where y^* denotes the output corresponding to the second intersection of $S(y)$ and δk. Hence a single country—with a fixed savings function, deprecation rate and production function, may end up with different long run income levels, depending on the initial level of capital available to it. The situation is represented in Figure 4. In this case economies to the left of k_* shrink while economies between k_* and k^* grow.



Politics. Beyond the roles for **government** already mentioned, there is in this model a role for governments to support the economy by accessing sources of capital, through aid, debt or foreign investment. Under some interpretations, all that is needed is that the government opens up capital markets.

4 A Simple Endogenous Growth Model

The previous model differed from the Solow model in that dynamics could push a middle income country into poverty, and the equilibrium point reached depends on where the economy starts. It shares with it a feature that growth at the steady state depends on exogenous factors—such as exogenous changes to the production function or the economy's endowment of capital. Recent work in growth theory—notably work on endogenous growth models—has tried to get beyond this. These models are marked by a more micro-level approach in which aggregate production is a function of the actions of individual producers, as well as by a greater concern with the role of information and knowledge.

We now consider a very simple model from Barro and Sala-i-Martin. In this model, output for firm i is given by:

$$Y_i = A(K_i)^\alpha (KL_i)^{1-\alpha}$$

Where K is the average amount of capital employed in the economy. The core idea here is that the effectiveness of labor in this model depends on the society-wide (average) level of capital, K, and not just on K_i, the private capital available to firm i. Dividing across by the number of employees in the firm, and by average capital per worker, we have:

$$\frac{\frac{Y_i}{L_i}}{\frac{K}{L}} = A\left(\frac{K_i}{L_i}\frac{L}{K}\right)^\alpha (L)^{1-\alpha}$$

$$\frac{y_i}{k} = A\left(\frac{k_i}{k}\right)^\alpha (L)^{1-\alpha}$$

In equilibrium (where all firms act in the same way and so $k_i = k$), we have:

$$\frac{y}{k} = AL^{1-\alpha}$$

$$y = AL^{1-\alpha}k$$

In this case, and contrary to what we saw above in our discussion of the Solow model, *output per unit capital* does **not** depend on the amount of capital in the economy; furthermore it is larger the larger is the economy-wide workforce. This implies that the break on capital growth—due in the Solow model to diminishing private marginal returns of capital—does not apply here.

What is happening in the model is that each firm is capturing only a share of the total benefits of their use of capital. The rest of the benefits go to other firms. While individual firms do not take into account the benefits of their capital investments for other firms, these nevertheless aggregate up. The reason why the size of the labor force is so important in this model is that it increases the productivity of capital for firms, and the more capital they employ, the more all benefit.

Politics. In this model we have that firms under-invest relative to the benefits that their investments bring to the economy as a whole. Policies that increase private investment will therefore have beneficial effects in this model.

5 A simple model of redistribution and growth

A lot of the work we will look at in later weeks will examine the relationship between redistribution and growth. At the heart of these models is the idea that redistribution often comes at a cost. If one person tries to redistribute from person 1 to person 2, person 2 might take actions to prevent that redistribution; but these actions put a constraint on person 2's economic activity and result in lower production than would otherwise be the case.

This core logic is perhaps most simply illustrated by considering a producer who can divide her unit of labor time into production of two goods in the knowledge that some other group (the "elite", "looters", "government") will try to extract some of her production. Say that there is a lootable good that exhibits constant returns and a non-lootable good that exhibits diminishing returns. The producer chooses a level of x to go to producing the lootable good; the rest goes for the non-lootable good.

In particular assume that:

$$y = \alpha x + \ln(1 - x)$$

and so utility is:

$$u_{producer} = (1 - \beta)\alpha x + \ln(1 - x)$$

Where $\alpha > 1$ describes the returns to the lootable sector and β is the share of production in the sector that the producer expects to be looted. Diminishing returns to the non lootable sector is captured by $\ln(1 - x)$.

First order conditions are then:

$$(1 - \beta)\alpha - \frac{1}{1 - x^*} = 0$$

$$x^* = 1 - \frac{1}{(1 - \beta)\alpha}$$

(For now lets assume that β is such that this is positive, later we will see that it is) The total value of production is maximized with $x = 1 - \frac{1}{\alpha}$ and so for any $\beta > 0$ there is too little effort put into the lootable sector. Equivalently, the larger is β the more time is allocated to the non-lootable good and the lower overall is the value of production.

Consider now the looter (say "the elite"). The looter wants to choose a policy β that maximizes his loot. But he knows that if β is too high there will be nothing there to loot! The looter then maximizes:

$$u_{elite} = \beta\alpha\left(1 - \frac{1}{(1-\beta)\alpha}\right)$$

The first order condition is given by

$$\alpha\left(1 - \frac{1}{(1-\beta^*)\alpha}\right) - \beta^*\alpha\left(\frac{1}{(1-\beta^*)^2\alpha}\right) = 0$$

Which is solved by

$$\beta^* = 1 - \alpha^{-.5}$$

In equilibrium the allocation to the lootable sector is then:

$$x^* = 1 - \frac{1}{(1-(1-\alpha^{-.5}))\alpha} = 1 - \alpha^{-.5}$$

The more productive the lootable sector the greater the rate of predation. The most important idea illustrated here is the following: we often think of redistribution in terms of simple transfers from one person to another, but in situtations when individuals with political power attempt to redistribute away from individuals with economic power, redistributon comes at a cost. In these cases, policy makers may take choices that increase their share of the pie while at the same time reducing the total size of the pie.

6 Inequality, Taxation and Investment

A different approach, found in Persson and Tabellini (Chapter 14.1), uses more detailed political economy aspects—in this case including a population of heterogenous citizens that can select government taxation and expenditure behavior—but a simpler macroeconomic set-up. In this case the authors provide a simple 2 period model with highly specific functional forms.

The model that follows is designed to focus on the incentives to invest and accumulate capital given that political processes set government tax and expenditure policies that can alter the incentives and performance of firms.

Let there be a continuum of agents, who differ in the size of their holdings of capital: let each have an endowment of $1 + e^i$, where 1 is the average endowment, and deviations from the average, e^i, are distributed according to $F(e^i)$ (e_i has 0 mean).

In Period 1:

- Citizens vote over a lump sum (poll) tax, t and a corporate tax τ. They can choose any level of taxation, using majority rule.
- Government uses t to improve the productivity of second period captial according to $A(t)$.
- The citizens choose how to divide their post-tax endowment between investments for the second period (k^i) and consumption (c_1^i). Hence $1 + e^i = t + c_1^i + k_i$
- The result of these choices is k —the amount of capital saved (per capita) for period 2.

In Period 2:

- $A(t)k$ is the per capita GDP, where $A(t)$ is increasing and concave in t (government's investment in period 1). k is simply the average value of k_i.
- The government uses a corporate tax τ on second period production to finance public expenditures, g, in the second period. Hence: $g = \tau A(t)k$.
- The citizens consume whatever they produce, less the government's tax on their produce: $c_2^i = (1 - \tau)A(t)k_i$
- The citizens also enjoy some public good produced by government to the value of $h(g)$

Individual i has utility:

$$V^i = \ln(c_1^i) + c_2^i + h(g)$$

Substituting for consumption (c_1^i and c_2^i,) this is:

$$V^i = \ln(1 + e^i - t - k_i) + (1 - \tau)A(t)k_i + h(\tau A(t)k)$$

From the point of view of capital accumulation and growth, the key decision is how much capital, k_i, to shift from period 1 to period 2.

The first order conditions for maximizing V^i (with respect to k_i) are:

$$\begin{aligned} -\frac{1}{1 + e^i - t - k_i*} + (1 - \tau)A(t) &= 0 \\ &\leftrightarrow \\ k^{i*} &= 1 + e^i - t - \frac{1}{(1 - \tau)A(t)} \end{aligned} \tag{7}$$

(Note that we have assumed that there are sufficiently many players that k is unaffected by the choice of any k_i.)

First period consumption is then given by $c_1^{i*} = 1 + e^i - t - k^{i*} = \frac{1}{(1-\tau)A(t)}$, which is independent of e^i.

At its maximum then, an agent's utility is given by:

$$V^i = \ln\left(\frac{1}{(1-\tau)A(t)}\right) + (1-\tau)A(t)[1 + e^i - t - \frac{1}{(1-\tau)A(t)}] + h(\tau A(t)k)$$

or

$$\begin{aligned} V^i \;&=\; \left[(1-\tau)A(t)e^i\right] \\ &\quad + \left[\ln\left(\frac{1}{(1-\tau)A(t)}\right) - (1-\tau)A(t)\left(t + \frac{1}{(1-\tau)A(t)} - 1\right)\right] \\ &\quad + [h(\tau A(t)k)] \end{aligned}$$

Written this way, we see that there are three parts to utility. The first part in square brackets is the idiosyncratic part, it is the benefit that each player gets in the second period from investing the rest of her endowment (net of taxes and period 1 consumption); it is increasing in the individual's endowment. The second part in square brackets is common to all individuals and represents common consumption in both periods. The third term is also common, but represents benefits from the production of the public good in the second period.

OK. Now let's think about what "optimal" choices of t and τ would be. Making use of the fact that the average value of e^i is 0, a social welfare maximizer's indirect utility function over consumption (maximizing average utility of consumption goods) is given by:

$$W(t,\tau) = \ln\left(\frac{1}{(1-\tau)A(t)}\right) - (1-\tau)A(t)\left(t + \frac{1}{(1-\tau)A(t)} - 1\right) + h(\tau A(t)k)$$

The social welfare maximizer's first order conditions are then:

$$\frac{\partial W}{\partial t} = 0$$

and

$$\frac{\partial W}{\partial \tau} = 0$$

We assume that h is such that the second order conditions are satisfied and that there is a unique solution to the first order conditions that gives a global maximum.

Now let's turn to voters. We can now write voter utility as:

$$V^i = (1-\tau)Ae^i + W$$

Consider now the choice of τ and t. First order conditions for maximizing the median voter's utility[2] with respect to these policy instruments are:

$$(1-\tau)A_t e^{median} + \frac{\partial W}{\partial t} = 0$$

and

$$-Ae^{median} + \frac{\partial W}{\partial \tau} = 0$$

Clearly these can differ from the policy choices of the welfare maximizer.

In particular, in the presence of inequality, median income can be much less than mean income. In this case e^{median} is negative and the median voter will choose (t,τ) such that $\frac{\partial W}{\partial t}$ is greater and $\frac{\partial W}{\partial \tau}$ is less than they would be for a social welfare maximizer. This is achieved by selecting a *lower* value of t, and a *higher* value of τ than would the social welfare maximizer.[3]

The intuition for the result is that below average capital owners bring less capital into the second period, and gain less from the increased productivity in the second period arising from a high level of taxation, t. They are therefore *less* supportive of raising t, that is they are less likely to support taxation-for-investment. They do gain however from the production of public goods that are consumed directly, via τ, but since these are financed by a capital tax, raising τ hurts them less than it does the average capital owner. They therefore support higher levels of τ, taxation-for-consumption. The net effect is that insofar as growth is increasing in t and decreasing in τ (see Equation 7), it is lower than it would be under the social welfare maximizer's choice.

This models then links national income to economic policy choices under democratic rule as a function of inequality. In the model democratic decision making has an adverse effect on national income in the presence of inequality (the results are reversed however whenever median income exceeds mean income).

[2]It is enough to look at the preferences of the median voter—even though there are two policy issues—because each player's indirect utility over outcomes is linear in her idiosyncratic feature (e^i).

[3]Establishing this formally requires a thorough analysis of the properties of W and h. However, the main point can be seen by noting that with e^i negative, the main difference in welfare between the median and the optimizer, given by $(1-\tau)Ae^{median}$, is increasing in τ and hence the median gets more out of τ than the optimizer does; but $(1-\tau)Ae^{median}$ is decreasing in A (remember, $e^{median} < 0$) and hence in t. The median then gets less benefits from t than does the welfare maximizer and so chooses a lower level.